ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 177–201

THE HARDY–LITTLEWOOD PROPERTY OF BANACH LATTICES

BY

J. GARCIA-CUERVA*

Departamento de Matemáticas
Módulo C-XVI, Facultad de Ciencias
Universidad Autónoma, 28049 Madrid, Spain

AND

R. MACIAS**

Universidad Nacional del Litoral
Conicet, Güemes 3450
3000 Sante Fé, Argentina

AND

J. L. TORREA*

Departamento de Matemáticas
Módulo C-XVI, Facultad de Ciencias
Universidad Autónoma, 28049 Madrid, Spain

ABSTRACT

We study a property of a Banach lattice which is charaterized by the boundedness in several classical spaces of a version of the Hardy–Littlewood maximal function obtained by taking the supremum of averages in the order of the lattice. This property is related to the well known U.M.D. condition.

* The first and third authors were partially supported by DGICYT, Spain, under Grant PB 86-108.

** The second author was supported by Ministerio de Educación, Spain, under a Sabbatical Grant (1989).

Received July 25, 1991 and in revised form May 28, 1992

Introduction

The motivation for this paper arises from a result of Bourgain [2] (see also [14]) which characterizes the U.M.D. property of a Banach function space in terms of a version $\mathcal{M}$ of the Hardy–Littlewood maximal function.

If X is a Banach space of functions on a given measure space Ω and if we view a function $f\colon \mathbf{R}^n \to X$ as a function of two variables $f(x,\omega), x \in \mathbf{R}^n, \omega \in \Omega$, $\mathcal{M}$ is just the ordinary Hardy–Littlewood maximal operator acting on the variable x. Then Bourgain's characterization says that X is U.M.D. if and only if $\mathcal{M}$ is bounded in $L^p_X(\mathbf{R}^n)$ and also in $L^{p'}_{X'}(\mathbf{R}^n)$ for some $p, 1 < p < \infty$, where p' is the exponent conjugate to p and X' is the function space dual.

We can also view $\mathcal{M}$ as a supremum of averages, but a supremum in the lattice. We adopt this point of view and we study those Banach lattices X for which $\mathcal{M}$ is bounded in $L^p_X(\mathbf{R}^n)$ for some $p, 1 < p < \infty$.

We call this property the Hardy–Littlewood (H.L.) property. Actually since we consider general lattices, our definition is slightly more complicated because we are forced to consider suprema of finite families only.

The main idea is that the operator $\mathcal{M}$ has a smooth version that can be viewed as a vector-valued singular integral. In section 1 we use the general theory of vector-valued singular integrals (as in [15]) to obtain characterizations of the H.L. property in terms of the boundedness of $\mathcal{M}$ in different function spaces associated to the lattice X. In section 2 we present several examples of lattices having or not having the H.L. property. It turns out that both ℓ^∞ and c_0 have the property H.L. while ℓ^1 does not have it. We also see that some convexity is necessary in order to have the H.L. property.

In section 3 we use the operator $\mathcal{M}$ (or its smooth version) to define new Hardy spaces and also a B.M.O. associated to the lattice X and we study the relation between the new spaces and the standard ones.

Finally in section 4 we take up several questions which are meaningful when one works in the torus instead of $\mathbf{R}^n$. For example we see that even though L^1 does not have the H.L. property, however $\mathcal{M}$ is bounded from $L^p_{L^1}(\mathbf{T})$ to $L^p_{L^\alpha}(\mathbf{T})$ for $1 < p < \infty$ and $0 < \alpha < 1$. This extends a result of Bourgain for the Hilbert transform (see [4]).

We want to thank Felipe Zo for many interesting conversations concerning section 4.

1. Main results

By a **Banach lattice** we shall mean a Banach space X over the field $\mathbf{R}$ of the real numbers, together with an order relation $\leq$ on X, satisfying the following properties:

(i) $x \leq y$ implies $x+z \leq y+z$ for every $x, y, z \in X$.

(ii) $ax \geq 0$ for every $x \geq 0$ in X and every $a \geq 0$ in $\mathbf{R}$.

(iii) for every $x, y \in X$, there exists the least upper bound $\sup\{x, y\}$ and also the greatest lower bound $\inf\{x, y\}$, and

(iv) if $|x|$ is defined as $|x| = \sup\{x, -x\}$, then the order relation $|x| \leq |y|$ implies the inequality between the norms $\|x\| \leq \|y\|$.

Whenever it is important to distinguish between the norms in different Banach spaces, we shall denote the norm in X by $\| \ \|_X$.

Definition 1.1: Let X be a Banach lattice and let J be a finite subset of the set $\mathbf{Q}_+$ of the positive rational numbers. Given a locally integrable function $f\colon \mathbf{R}^n \to X$ (this means, of course, a strongly measurable f such that the scalar function $y \longmapsto \|f(y)\|_X$ is locally integrable) we define:

$$\mathcal{M}_J f(x) = \sup_{r \in J} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

where $|B(x,r)| = c_n r^n$ is the Lebesgue measure of the ball $B(x,r)$. ∎

We shall always denote the Lebesgue measure of a set E by $|E|$.

Notice that the sup in definition 1.1 is a sup in the lattice X. This accounts for the need to take just a finite collection of radii J. This difficulty will disappear when we deal with the most relevant examples, which will turn out to be order complete (see the remark below).

The family $\{\mathcal{M}_J\}$ where J ranges over all finite subsets of $\mathbf{Q}_+$, will be our main object of study. We shall investigate the boundedness of $\mathcal{M}_J$ and related operators in the Bochner-Lebesgue spaces $L^p_X(\mathbf{R}^n)$, $1 < p \leq \infty$, the Lorentz space weak-$L^p_X(\mathbf{R}^n)$, the Hardy space $H^1_X(\mathbf{R}^n)$ and the space B.M.O.$_X(\mathbf{R}^n)$. For the definitions of these spaces, we refer the reader to [15], where one can find a complete account of the theory of vector-valued singular integrals, which will be the basic tool in what follows. Sometimes we shall need to consider the analogues of the spaces listed above for the case of finite measure, i.e. for $\mathbf{T}^n$, the n-dimensional torus. We shall also have to deal with the weighted spaces $L^p_X(w)$

where w is a weight in the class A_∞ of Muckenhoupt. These spaces appear also in [15] and the theory of Muckenhoupt weights can be seen in [6] and [10].

Definition 1.2: We shall say that a Banach lattice X satisfies the Hardy–Littlewood (H.L.) property if there exists some p_0, $1 < p_0 < \infty$ such that the operators $\mathcal{M}_J$ are uniformly bounded in $L_X^{p_0}(\mathbf{R}^n)$, that is, the inequality:

$$\|\mathcal{M}_J f\|_{L_X^{p_0}(\mathbf{R}^n)} \leq C\|f\|_{L_X^{p_0}(\mathbf{R}^n)}$$

holds with C independent of J. ∎

Remark 1.3: The H.L. property does not depend on the dimension n.

One can dominate $\mathcal{M}_J$ (in the lattice order) by an average over a hemisphere of the corresponding one dimensional operators in different directions, exactly as one does in the method of rotations (see [8], p. 79 or [10], p. 223). That way we prove that if X has the H.L. property with $n = 1$, then it has it also with any other n.

The converse is even easier, perhaps passing through the corresponding property in finite measure ($\mathbf{T}^n$), which turns out to be equivalent as well. We shall keep n fixed with the understanding that its particular value is irrelevant. ∎

Remark 1.4: Let $(\Omega, \sum, \mu)$ be a complete σ-finite measure space. A Banach space X consisting of equivalence classes, modulo equality almost everywhere (a.e.), of locally integrable, real valued functions on Ω is called a Köthe function space if the following two conditions hold:

(1) If $|f(\omega)| \leq |g(\omega)|$ a.e. on Ω with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$.
(2) For every $E \in \sum$ with $\mu(E) < \infty$, the characteristic function $\mathcal{X}_E$ of E belongs to X.

For the main facts on Banach lattices and Banach function spaces, we refer the reader to [13], whose terminology and notation we shall adopt. Another useful reference is [1].

Every Köthe function space is a Banach lattice with the obvious order ($f \geq 0$ if $f(\omega) \geq 0$ for a.e. ω).

If X is a Köthe function space and $f\colon \mathbf{R}^n \to X$ is a locally integrable function, it is clear that $\mathcal{M}_J f(x)$ is a function of ω given by:

$$\mathcal{M}_J f(x)(\omega) = \sup_{r \in J} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y,\omega)|dy$$

where sup is now the sup in the order of $\mathbf{R}$. In this situation we can see f and $\mathcal{M}_J f$ as functions on $\mathbf{R}^n \times \Omega$.

Moreover, in this case we can define the operator $\mathcal{M}$ given by:

$$\mathcal{M}f(x,\omega) = \sup_{r\in\mathbf{Q}_+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y,\omega)|dy. \tag{1.5}$$

We recall that a Köthe function space is said to have the Fatou property (see [13]) if everytime we have a sequence of functions $f_n \in X$, such that $f_n(\omega) \geq 0$ for a.e. ω, $f_n(\omega) \uparrow f(\omega)$ for a.e. ω and also $\sup_n \|f_n\| < \infty$, then we have $f \in X$ and $\|f\| = \lim\limits_n \|f_n\|$.

It is a simple consequence of Lebesgue's monotone convergence theorem for scalar functions that the space $L^{p_0}_X(\mathbf{R}^n)$ has the Fatou property provided X has it. Therefore, *a Köthe function space having the Fatou property satisfies the H.L. property if and only if there exists some $p_0, 1 < p_0 < \infty$ such that $\mathcal{M}$ is bounded in $L^{p_0}_X(\mathbf{R}^n)$.* ∎

We shall need to consider as auxiliary operators, smooth versions of $\mathcal{M}_J$ and $\mathcal{M}$ which we define next:

Definition 1.6: Let φ: $[0,\infty[\to \mathbf{R}$ be a smooth function such that

$$\mathcal{X}_{[0,1]}(t) \leq \varphi(t) \leq \mathcal{X}_{[0,2]}(t)$$

for every $t \geq 0$.

Let X be a Banach lattice, J a finite subset of $\mathbf{Q}_+$ and f: $\mathbf{R}^n \to X$ locally integrable. We define

$$\mathcal{M}_{\varphi,J}f(x) = \sup_{r\in J}\left|\frac{1}{c_0 r^n}\int_{\mathbf{R}^n}\varphi\left(\frac{|x-y|}{r}\right)f(y)dy\right|,$$

$x \in \mathbf{R}^n$, where $c_0 = \int_{\mathbf{R}^n}\varphi(|x|)dx$.

If X is a Köthe space of functions on Ω, we can also define:

$$\mathcal{M}_{\varphi}f(x,\omega) = \sup_{r\in\mathbf{Q}_+}\left|\frac{1}{c_0 r^n}\int_{\mathbf{R}^n}\varphi\left(\frac{|x-y|}{r}\right)f(y,\omega)dy\right|.$$

In the definition of the $\mathcal{M}_{\varphi,J}$'s, the sup and the $|\ |$ are those in X, while in the definition of $\mathcal{M}_\varphi$ they are the corresponding ones in $\mathbf{R}$. ∎

The theory of vector-valued singular integrals can be applied to obtain the following list of different characterizations of the H.L. property.

THEOREM 1.7: *Given a Banach lattice X and a function φ as in the previous definition, the following conditions are equivalent:*

(1) *X has the H.L. property.*

(2) *There exists some $p_0, 1 < p_0 < \infty$, such that*

$$\|\mathcal{M}_{\varphi,J} f\|_{L_X^{p_0}(\mathbf{R}^n)} \leq C_{p_0} \|f\|_{L_X^{p_0}(\mathbf{R}^n)}.$$

(3) *For every $p, 1 < p < \infty$*

$$\|\mathcal{M}_J f\|_{L_X^p(\mathbf{R}^n)} \leq C_p \|f\|_{L_X^p(\mathbf{R}^n)}.$$

(4) *For every $p, 1 < p < \infty$*

$$\|\mathcal{M}_{\varphi,J} f\|_{L_X^p(\mathbf{R}^n)} \leq C_p \|f\|_{L_X^p(\mathbf{R}^n)}.$$

(5) $|\{x \in \mathbf{R}^n : \|\mathcal{M}_J f(x)\|_X > \lambda\}| \leq \dfrac{C}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_X dx.$

(6) $|\{x \in \mathbf{R}^n : \|\mathcal{M}_{\varphi,J} f(x)\|_X > \lambda\}| \leq \dfrac{C}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_X dx.$

(7) $\|\mathcal{M}_{\varphi,J} f\|_{L_X^1(\mathbf{R}^n)} \leq C \|f\|_{H_X^1(\mathbf{R}^n)}.$

(8) $\|\mathcal{M}_{\varphi,J} f\|_{B.M.O._X(\mathbf{R}^n)} \leq C \|f\|_{L_X^\infty(\mathbf{R}^n)}.$

(9) *If w is an A_∞ weight in $\mathbf{R}^n$ and $0 < p < \infty$,*

$$\int_{\mathbf{R}^n} \|\mathcal{M}_{\varphi,J} f(x)\|_X^p w(x) dx \leq C_p(w) \int_{\mathbf{R}^n} (M(\|f\|_X)(x))^p w(x) dx,$$

where M is the Hardy–Littlewood maximal operator in $\mathbf{R}^n$, which is applied to the scalar function $x \longmapsto \|f(x)\|_X$.

(10) *For every cube Q (with sides parallel to the coordinate axes, as we shall always assume) and every function $f \in L_X^\infty(\mathbf{R}^n)$ having support contained in Q (supp $f \subset Q$) we have:*

$$\int_Q \|\mathcal{M}_{\varphi,J} f(x)\|_X dx \leq C \|f\|_{L_X^\infty} |Q|.$$

The constants $C, C_p, C_p(w)$ (not the same at each occurrence) depend on X, φ, p or w, but do not depend on J.

Proof: Observe that for every $r > 0$, we have:

$$\mathcal{X}_{B(x,r)}(y) \leq \varphi\left(\frac{|x-y|}{r}\right) \leq \mathcal{X}_{B(x,2r)}(y); \qquad x, y \in \mathbf{R}^n$$

and, consequently, if $f\colon \mathbf{R}^n \to X$ is a positive function:

$$\mathcal{X}_{B(x,r)}(y)f(y) \leq \varphi\left(\frac{|x-y|}{r}\right) f(y) \leq \mathcal{X}_{B(x,2r)}(y)f(y)$$

so that we get:

$$\mathcal{M}_J f(x) \leq \frac{c_0}{c_n}\mathcal{M}_{\varphi,J}f(x), \qquad x \in \mathbf{R}^n$$

and

$$\mathcal{M}_{\varphi,J}f(x) \leq \frac{2^n c_n}{c_0}\mathcal{M}_{2J}f(x), \qquad x \in \mathbf{R}^n.$$

These inequalities, together with the fact that, for any function $f\colon \mathbf{R}^n \to X$, we have

$$\||f(x)|\|_X = \|f(x)\|_X$$

immediately yield (1) $\Longleftrightarrow$ (2), (3) $\Longleftrightarrow$ (4) and (5) $\Longleftrightarrow$ (6).

Next we shall prove that (2) $\Rightarrow$ (6).

We just need to consider one fixed J and see that the boundedness of $\mathcal{M}_{\varphi,J}$ in $L_X^{p_0}$ implies that it is bounded from L_X^1 to weak-L_X^1 with a constant depending only on its norm as an operator bounded in $L_X^{p_0}$.

In order to do that, we shall consider an operator T_J sending X-valued functions into functions taking values in the Banach space $X(J)$, consisting of the sequences $(x_r)_{r\in J}$ of elements $x_r \in X$ with $\|(x_r)_{r\in J}\|_{X(J)} = \|\sup|x_r|\|_X$.

For $f\colon \mathbf{R}^n \to X$ locally integrable, we define:

$$T_J f(x) = \left(\frac{1}{c_0 r^n}\int_{\mathbf{R}^n}\varphi\left(\frac{|x-y|}{r}\right)f(y)dy\right)_{r\in J}.$$

This is to be viewed as a linearization of the operator $\mathcal{M}_{\varphi,J}$. Since

$$\|T_J f(x)\|_{X(J)} = \|\mathcal{M}_{\varphi,J}f(x)\|_X$$

the boundedness of $\mathcal{M}_{\varphi,J}$ in $L_X^{p_0}(\mathbf{R}^n)$ is equivalent to the boundedness of T_J from $L_X^{p_0}(\mathbf{R}^n)$ to $L_{X(J)}^{p_0}(\mathbf{R}^n)$.

But T_J is a linear operator given by convolution with a kernel $K_J(x) \in \mathcal{L}(X, X(J))$ (bounded linear operators from X to $X(J)$) namely:

$$K_J(x)v = \left(\frac{1}{c_0 r^n}\varphi\left(\frac{|x|}{r}\right)v\right)_{r\in J}; \qquad v \in X,$$

$$\|K_J(x)\|_{\mathcal{L}(X,X(J))} = \sup_{r\in J} \frac{1}{c_0 r^n}\varphi\left(\frac{|x|}{r}\right).$$

The smoothness of φ guarantees that K_J satisfies the so-called standard estimates for Calderón–Zygmund kernels, that is:

$$\|K_J(x)\| \le C|x|^{-n}$$

and

$$\|K_J(x) - K_J(x')\| \le C|x-x'||x|^{-n-1}.$$

Note that these estimates are uniform in J, i.e., the constant C does not depend on J. Now the theory of vector-valued Calderón–Zygmund operators, as given in [15] can be applied to T_J to obtain

$$|\{x\in \mathbf{R}^n\colon \|T_Jf(x)\|_{X(J)} > \lambda\}| \le \frac{C}{\lambda}\int_{\mathbf{R}^n} \|f(x)\|_X\,dx \tag{1.8}$$

But this is precisely (6)

Our next step is to see that (6) $\Rightarrow$ (9).

Given J fixed we consider the $X(J)$-valued operators

$$T_{J,\varepsilon}f(x) = \left(\frac{1}{c_0r^n}\int_{|x-y|>\varepsilon}\varphi\left(\frac{|x-y|}{r}\right)f(y)dy\right)_{r\in J}, \qquad \varepsilon>0$$

and the corresponding maximal operator

$$T_J^*f(x) = \sup_{\varepsilon>0}\|T_{J,\varepsilon}f(x)\|_{X(J)}.$$

Now for this operator we have the following Cotlar's inequality:

$$T_J^*f(x) \le C_\delta\{(M(\|T_Jf\|^\delta_{X(J)})(x))^{1/\delta} + M(\|f\|_X)(x)\} \tag{1.9}$$

valid for $0<\delta\le 1$.

Inequality (1.9) is obtained exactly as in [11], p. 56, by using just the weak type (1.1) of T_J, (that is, (1.8) which is equivalent to (6)) plus the standard estimates.

Now it is a simple consequence of (1.8) and Kolmogorov's inequality (see [11], p. 5 or [10], p. 485) that the operator $f\longmapsto (M(\|T_Jf\|^\delta_{X(J)}))^{1/\delta}$ for $0<\delta<1$ is bounded from $L^1_X(\mathbf{R}^n)$ to weak-$L^1(\mathbf{R}^n)$. Therefore, (1.9) implies that:

$$|\{x\in\mathbf{R}^n\colon T_J^*f(x)>\lambda\}| \le \frac{C}{\lambda}\int_{\mathbf{R}^n}\|f(x)\|_X\,dx. \tag{1.10}$$

Now from (1.10) we derive, just as in [6] the good-λ inequality

$$\begin{aligned} &w(\{x \in \mathbf{R}^n : T_J^* f(x) > 2\lambda \text{ and } M(\|f\|_X)(x) \le \gamma\lambda\}) \\ &\le C\gamma^\delta w(\{x \in \mathbf{R}^n : T_J^* f(x) > \lambda\}) \end{aligned}$$

for $w \in A_\infty$. This good-λ inequality produces, in the usual way (see [6]) the inequality:

$$\int_{\mathbf{R}^n} (T_J^* f(x))^p w(x)dx \le C \int_{\mathbf{R}^n} (M(\|f\|_X)(x))^p w(x)dx \tag{1.11}$$

for $w \in A_\infty$ and $0 < p < \infty$. Of course C depends on p and w, $C = C_p(w)$. In order to obtain 9) from (1.11) we just need to observe that

$$\begin{aligned} &\|T_{J,\varepsilon} f(x) - T_J f(x)\|_{X(J)} \\ &\le \int_{\mathbf{R}^n} \|K_J(x-y)\mathcal{X}_{B(x,\varepsilon)}(y) f(y)\|_{X(J)} dy \\ &= \int_{\mathbf{R}^n} \sup_{r\in J} \left| \frac{1}{c_0 r^n}\varphi\left(\frac{|x-y|}{r}\right)\right| \|f(y)\|_X \mathcal{X}_{B(x,\varepsilon)}(y) dy \\ &\le \frac{C}{(\min J)^n} \int_{B(x,\varepsilon)} \|f(y)\| dy \to 0, \qquad \varepsilon \to 0 \end{aligned}$$

so that we can use Fatou's lemma to obtain:

$$\begin{aligned} &\int_{\mathbf{R}^n} \|\mathcal{M}_{\varphi,J} f(x)\|_X^p w(x)dx = \int_{\mathbf{R}^n} \|T_J f(x)\|_{X(J)}^p w(x)dx \\ &\le \int_{\mathbf{R}^n} (T_J^* f(x))^p w(x)dx \le C \int_{\mathbf{R}^n} (M(\|f\|_X)(x))^p w(x)dx \end{aligned}$$

and we get (9).

It is immediate that (9) implies (4). We just need to take $w \equiv 1$ and use the boundedness of M in L^p for $p > 1$.

Since (4) trivially implies (2), we have obtained the equivalence of (1), (2), (3), (4), (5), (6) and (9). It is rather simple to obtain (10) from (2).

Let f be a function in L_X^∞ with supp $f \subset Q$. Then

$$\begin{aligned} &\frac{1}{|Q|}\int_Q \|\mathcal{M}_{\varphi,J} f(x)\|_X dx \le \left(\frac{1}{|Q|}\int_Q \|\mathcal{M}_{\varphi,J} f(x)\|_X^{p_0} dx\right)^{1/p_0} \\ &\le C\left(\frac{1}{|Q|}\int_Q \|f(x)\|_X^{p_0} dx\right)^{1/p_0} \le C\|f\|_{L_X^\infty}. \end{aligned}$$

Now, assuming (10), we shall derive (8).

Let $f \in L^\infty_X$. Given a cube Q with center x_0, we consider the cube $\tilde{Q}$ with the same center and diameter twice that of Q, and decompose $f = f_1 + f_2$ where $f_1 = f\mathcal{X}_{\tilde{Q}}$. Then we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \|\mathcal{M}_{\varphi,J}f(x) - \mathcal{M}_{\varphi,J}f_2(x_0)\|_X\,dx \\
&\leq \frac{1}{|Q|}\int_Q \|T_Jf(x) - T_Jf_2(x_0)\|_{X(J)}dx \\
&\leq \frac{1}{|Q|}\int_Q \|T_Jf_1(x)\|_{X(J)}dx + \frac{1}{|Q|}\int_Q \|T_Jf_2(x) - T_Jf_2(x_0)\|_{X(J)}dx.
\end{aligned}$$

The first term in this sum is handled by using (10):

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \|T_Jf_1(x)\|_{X(J)}dx = \frac{1}{|Q|}\int_Q \|\mathcal{M}_{\varphi,J}f_1(x)\|_X\,dx \\
&\leq \frac{2^n}{|\tilde{Q}|}\int_{\tilde{Q}} \|\mathcal{M}_{\varphi,J}f_1(x)\|_X\,dx \leq C\|f\|_{L^\infty_X}.
\end{aligned}$$

For the second term we just need to use the standard estimates for K_J. We get:

$$\begin{aligned}
\|T_Jf_2(x) - T_Jf_2(x_0)\|_{X(J)} &\leq C\int_{\mathbf{R}^n} \frac{|x-x_0|}{|x-x_0|^{n+1} + |y-x_0|^{n+1}}\|f(y)\|_X\,dy \\
&\leq C\|f\|_{L^\infty_X}.
\end{aligned}$$

That way we obtain (8).

We have actually found that T_J is bounded from L^∞_X to B.M.O. $_{X(J)}$.

Now from (8) one can obtain (7). We just need to take an atom $a \in H^1_X$ and show that:

$$\|\mathcal{M}_{\varphi,J}a\|_{L^1_X(\mathbf{R}^n)} = \|T_Ja\|_{L^1_{X(J)}(\mathbf{R}^n)} \leq C.$$

That a is an atom means that a: $\mathbf{R}^n \to X$ is supported in a cube Q, with center x_0, say; a has also average 0 and it satisfies $\|a(x)\|_X \leq 1/|Q|$ a.e. For $x \notin \tilde{Q}$, the doubled cube, we have:

$$\begin{aligned}
T_Ja(x) &= \int_Q K_J(x-y)a(y)dy \\
&= \int_Q (K_J(x-y) - K_J(x-x_0))a(y)dy
\end{aligned}$$

which, together with the standard estimates, gives:

$$\|\mathcal{M}_{\varphi,J}a(x)\|_X = \|T_J a(x)\|_{X(J)} \le C|x-x_0|^{-n-1}\int_Q |y-x_0|\|a(y)\|_X dy$$
$$\le C|Q|^{1/n}|x-x_0|^{-n-1}$$

and, consequently,

$$\int_{\mathbf{R}^n\setminus\tilde{Q}} \|\mathcal{M}_{\varphi,J}a(x)\|_X dx \le C.$$

On $\tilde{Q}$ we can use (8) which tells us that

$$\|\mathcal{M}_{\varphi,J}a\|_{B.M.O._X} \le C|Q|^{-1}.$$

In particular

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \|\mathcal{M}_{\varphi,J}a(x) - (\mathcal{M}_{\varphi,J}a)_{\tilde{Q}}\|_X dx \le C|Q|^{-1}.$$

But note also that if Q^* is a cube adjacent to $\tilde{Q}$ and with the same size

$$\|(\mathcal{M}_{\varphi,J}a)_{\tilde{Q}} - (\mathcal{M}_{\varphi,J}a)_{Q^*}\|_X \le C|Q|^{-1}$$

as one sees by adding and substracting the average on the smallest cube containing both $\tilde{Q}$ and Q^*.

Also, since $Q^* \subset \mathbf{R}^n \setminus \tilde{Q}$

$$\|(\mathcal{M}_{\varphi,J}a)_{Q^*}\|_X \le C|Q|^{-1},$$

so that, actually,

$$\|(\mathcal{M}_{\varphi,J}a)_{\tilde{Q}}\|_X \le C|Q|^{-1},$$

and

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \|\mathcal{M}_{\varphi,J}a(x)\|_X dx \le C|Q|^{-1}.$$

We get finally

$$\int_{\mathbf{R}^n} \|\mathcal{M}_{\varphi,J}a(x)\|_X dx = \int_{\tilde{Q}} + \int_{\mathbf{R}^n\setminus\tilde{Q}} \le C.$$

That (7) implies (10) is almost immediate. We just need to go from f to an atom

$$a(x) = (2\|f\|_{L^\infty_X}|Q|)^{-1}(f(x) - f_Q).$$

We have

$$\int_Q \|\mathcal{M}_{\varphi,J}f(x)\|_X dx \le \int_Q \|\mathcal{M}_{\varphi,J}(f-f_Q)(x)\|_X dx$$
$$+ |Q|\cdot\|f_Q\|_X \le 2\|f\|_{L^\infty_X}|Q|\|\mathcal{M}_{\varphi,J}a\|_{L^1_X}$$
$$+ |Q|\cdot\|f\|_{L^\infty_X} \le C\|f\|_{L^\infty_X}|Q|.$$

We have thus shown that (7), (8) and (10) are equivalent. On the other hand they imply (2). This is obtained by interpolation. Indeed, from (10) we get, after fixing J, that the linear operator T_J is bounded from H^1_X to $L^1_{X(J)}$ and also from L^∞_X to B.M.O.$_{X(J)}$. Interpolation can be applied in this setting to get (2). This finishes the proof of the theorem. ∎

2. Examples of lattices having and not having the Hardy–Littlewood property

A large class of examples is provided by those Köthe function spaces satisfying the condition known as U.M.D. (from "unconditional martingale differences"). Recall that a Banach space X is said to be U.M.D. if it satisfies an inequality

$$\|\sum_{k=1}^{n}\varepsilon_k d_k\|_{L^p_X} \le C_{p,X}\|\sum_{k=1}^{n} d_k\|_{L^p_X}$$

for all $n\in N$, $\varepsilon_k=\pm1$ and for all X-valued martingale differences $\{d_k\}_{k\ge1}$, where p is some exponent such that $1<p<\infty$ (see [5]).

It is a result of Bourgain [2] (see also [14]) that a Köthe function space X is U.M.D. if and only if the operator $\mathcal{M}$ defined in (1.5) is bounded in L^p_X and in $L^{p'}_{X'}$, where $1<p<\infty$, p' is the conjugate exponent and X' is the function space dual of X. As a consequence of Bourgain's result we get, in our terminology, the following.

Proposition 2.1: *For a Köthe function space X with the Fatou property, the following conditions are equivalent:*

(a) *X is U.M.D.*

(b) *Both X and X' satisfy the H.L. property.*

For a general Köthe function space, still (a) *implies* (b).

If, with the notation of remark 1.4, the underlying measure space Ω is the set of positive integers N, with μ the counting measure, we have, for

$$f(x)=(f_j(x))_j, \qquad \mathcal{M}f(x)=(Mf_j(x))_j.$$

Let us examine in this case, the behaviour of different sequence spaces with respect to the H.L. property.

We know that ℓ^p is U.M.D. provided $1 < p < \infty$. Thus, for this range ℓ^p is H.L.

This fact follows from the inequality

$$\|(\sum_{j=1}^{\infty} |Mf_j|^p)^{1/p}\|_{L^q(\mathbf{R}^n)} \le C_{p,q}\|(\sum_{j=1}^{\infty} |f_j|^p)^{1/p}\|_{L^q(\mathbf{R}^n)}. \tag{2.2}$$

We just need this inequality for some q, $1 < q < \infty$. Then theorem 1.7 tells us that it is valid for any $q, 1 < q < \infty$.

When $q < p$, (2.2) is simply a consequence of the fact that M is linearizable and positive (see [10], p. 482). For general q the inequality was obtained by Fefferman and Stein [9]. Actually (2.2) is true even for $p = \infty$, in the sense that

$$\|\sup_j Mf_j\|_{L^q(\mathbf{R}^n)} \le C_{p,\infty}\|\sup_j |f_j|\,\|_{L^q(\mathbf{R}^n)}. \tag{2.3}$$

This inequality is simply a consequence of the boundedness of M in $L^q(\mathbf{R}^n)$ since

$$\sup_j Mf_j(x) \le M(\sup_j |f_j|).$$

As a matter of fact, the way to obtain (2.2) in [10] for $q < p$, is to interpolate in p between $p = q$ and $p = \infty$ (i.e. (2.3)), interpolation being possible because M is linearizable.

Now (2.3) tells us that ℓ^∞ *has the H.L. property*. It also implies that c_0 *has the H.L. property* . Indeed we just need to observe that the subspace $\mathcal{F}$ consisting of those $f = (f_j)_j \in L^q_{c_0}(\mathbf{R}^n)$ such that all but a finite number of the components f_j vanish almost everywhere, is dense in $L^q_{c_0}(\mathbf{R}^n)$ and is obviously mapped into itself by $\mathcal{M}$, so that (2.3) implies

$$\|\mathcal{M}f\|_{L^q_{c_0}(\mathbf{R}^n)} \le C_{p,\infty}\|f\|_{L^q_{c_0}(\mathbf{R}^n)}.$$

Note that c_0 does not have the Fatou property, while ℓ^∞ does. The Fatou property is equivalent to having $X'' = X$ (see [13], p. 30) and we have $c_0'' = \ell^\infty$.

PROPOSITION 2.4: *ℓ^1 does not satisfy the H.L. property.*

Proof: We shall prove that $\mathcal{M}$ is not bounded in $L^p_{\ell^1}(\mathbf{R}), 1 < p < \infty$. According to remark 1.4 this is all that we need since ℓ^1 is a Köthe function space with

Fatou property. Let m be a fixed natural number and consider the function $F_m\colon \mathbf{R} \to \ell^1$ given by:

$$F_m(x) = (f_1(x), f_2(x), ..., f_m(x), 0, 0, ...)$$

where $f_j(x) = \mathcal{X}_{]\frac{j-1}{m},\frac{j}{m}]}(x), \qquad 1 \le j \le m.$

Then

$$\|F_m(x)\|_{\ell^1} = \sum_{j=1}^{m} f_j(x) = \mathcal{X}_{]0,1]}(x)$$

and

$$\|F_m\|_{L^p_{\ell^1}(\mathbf{R})} = 1.$$

On the other hand if $j \le k$ and $x \in [k-1/m, k/m]$, then

$$Mf_j(x) \ge \frac{1}{k-j+1}.$$

Therefore:

$$\begin{aligned}
\int_{\mathbf{R}} \|\mathcal{M}F_m(x)\|^p_{\ell^1}\,dx &= \int_{\mathbf{R}} (\sum_{j=1}^{m} Mf_j(x))^p\,dx \\
&\ge \sum_{k=1}^{m} \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(\sum_{j=1}^{m} Mf_j(x)\right)^p dx \\
&\ge \sum_{k=1}^{m} \frac{1}{m} \left(\sum_{j=1}^{k} \frac{1}{k-j+1}\right)^p \\
&= \sum_{k=1}^{m} \frac{1}{m} \left(1 + \frac{1}{2} + ... + \frac{1}{k}\right)^p \\
&\ge \frac{1}{m} \sum_{k=1}^{m} \left(1 + \frac{1}{2} + ... + \frac{1}{k}\right) \\
&= 1 + \frac{1}{2}\left(1 - \frac{1}{m}\right) + ... + \frac{1}{m}\left(1 - \frac{m-1}{m}\right) = A_m.
\end{aligned}$$

But $A_m \to \infty$ (note $A_m \ge C\sum_{j=1}^{m}(1/j)$) and the proof is complete. ∎

Definition 2.5: A Banach lattice X is said to be p-convex $1 \le p < \infty$, if the following inequality holds:

$$\|(\sum_{j=1}^{m}|x_j|^p)^{1/p}\|_X \le C_p(\sum_{j=1}^{m}\|x_j\|_X^p)^{1/p}$$

with a constant C_p independent of m. ∎

When X is a lattice of functions or, more generally, when X is order continuous, the concrete representation of the lattice allows us to define $(\sum_{j=1}^{m}|x_j|^p)^{1/p}$ in the obvious way. However, for a general lattice, these expressions need to be defined (see [13] 1.d). By combining theorems 1.f.12 and 1.f.7 of [13], we obtain the following useful result.

PROPOSITION 2.6: *Suppose X is a Banach lattice which is not p-convex for any $p > 1$. Then, for every $\varepsilon > 0$ and every positive integer m, there exists a sequence $\{e_i\}_{i=1}^{m}$ of pairwise disjoint elements in X (pairwise disjoint means that $\inf\{|e_i|,|e_j|\} = 0$, $i \neq j$) such that:*

$$(1-\varepsilon)\sum_{i=1}^{m}|a_i| \le \|\sum_{i=1}^{m}a_ie_i\|_X \le \sum_{i=1}^{m}|a_i| \tag{2.7}$$

for every choice of scalars $\{a_i\}$.

With the help of this proposition, we shall prove the following

THEOREM 2.8: *If a Banach lattice X satisfies the property H.L., then X is p-convex for some $p > 1$.*

Proof: Assume that X satisfies H.L. and also that X is not p-convex for any $p > 1$. From this we shall be able to show that ℓ^1 satisfies H.L. But this is a contradiction that proves the theorem. Let us see the details.

Let $\varepsilon > 0$ and $f(x) = (f_i(x))_i$ be a positive function $f \in L^p_{\ell^1}$. According to proposition 2.6, given m there exists a family $\{e_i\}_{i=1}^{m}$ of pairwise disjoint elements in X satisfying (2.7).

Let us consider $F_m(x) = (f_1(x),...,f_m(x),0,0,...)$. We shall make use of the following fact:

If $x_1,\dots,x_m$ are positive pairwise disjoint elements in X, then $\sum_{i=1}^{m}x_i = \sup_{1\le i\le m}\{x_i\}$.

If we just have two elements a and b this follows from:

$$\begin{aligned}0 = \inf\{a,b\} &= -\sup\{-a,-b\}\\ &= -(\sup\{-a+a+b,-b+a+b\}-(a+b))\\ &= a+b-\sup\{a,b\}.\end{aligned}$$

For the general case we use induction:

$$\begin{aligned}\sum_{i=1}^{m} x_i &= \left(\sum_{i=1}^{m-1} x_i\right) + x_m = \sup_{1\leq i\leq m-1}\{x_i\} + x_m \\ &= \sup_{1\leq i\leq m-1}\{x_i + x_m\} = \sup_{1\leq i\leq m-1}\{\sup\{x_i, x_m\}\} \\ &= \sup_{1\leq i\leq m}\{x_i\}.\end{aligned}$$

In the same fashion one can prove that for pairwise disjoint elements e_i and scalars a_i ,$|\sum_{i=1}^m a_i e_i| = \sum_{i=1}^m |a_i||e_i|$, so that in (2.7) we can assume that the elements e_i are positive.

Since

$$\sum_{i=1}^{m} f_i(x)e_i = \sup_{1\leq i\leq m}\{f_i(x)e_i\},$$

we have

$$\mathcal{M}_J(\sum_{i=1}^{m} f_i e_i)(x) = \sum_{i=1}^{m} M_J f_i(x) e_j$$

where we have written M_J for the operator $\mathcal{M}_J$ corresponding to the lattice $\mathbf{R}$. This and (2.7) allow us to relate the operators $\mathcal{M}_J$ of ℓ^1 and X, as follows:

$$\begin{aligned}\int_{\mathbf{R}^n} \|\mathcal{M}_J F_m(x)\|_{\ell^1}^p dx &= \int_{\mathbf{R}^n} (\sum_{i=1}^{m} M_J f_i(x))^p dx \\ &\leq \frac{1}{(1-\varepsilon)^p}\int_{\mathbf{R}^n} \|\sum_{i=1}^{m} M_J f_i(x) e_i\|_X^p dx \\ &= \frac{1}{(1-\varepsilon)^p}\int_{\mathbf{R}^n} \|\mathcal{M}_J(\sum_{i=1}^{m} f_i e_i)(x)\|_X^p dx \\ &\leq \frac{C(X,p)}{(1-\varepsilon)^p}\int_{\mathbf{R}^n} \|\sum_{i=1}^{m} f_i(x) e_i\|_X^p dx \\ &\leq \frac{C(X,p)}{(1-\varepsilon)^p}\int_{\mathbf{R}^n} \left(\sum_{i=1}^{m} f_i(x)\right)^p dx \\ &= \frac{C(X,p)}{(1-\varepsilon)^p}\int_{\mathbf{R}^n} \|F_m(x)\|_{\ell^1}^p dx.\end{aligned}$$

By letting $m \to \infty$ we would get that ℓ^1 has the H.L. property, which is a contradiction. ∎

Remark 2.9: It has to be noted that $\mathcal{M}$ is in general unbounded in L^∞_X. We shall give an example with $X = \ell^2$. Let $f(x) = (f_j(x))_{j=1}^\infty$ where $f_j = \mathcal{X}_{[2^{-j},2^{-j+1}]}$, $1 \le j < \infty$. Then $\|f(x)\|_{\ell^2} = \mathcal{X}_{]0,1]}(x)$ and consequently $f \in L^\infty_{\ell^2}$. But if $x \in [0, 2^{-N}], 0 < j \le N$, then $Mf_j(x) \ge 1/2$. Therefore

$$\|\mathcal{M}f(x)\|_{\ell^2} \ge (\sum_{j=1}^{N}(Mf_j(x))^2)^{1/2} \ge (1/2)N^{1/2}$$

so that $\mathcal{M}f \notin L^\infty_{\ell^2}$. ∎

Remark 2.10: It is a very simple fact that if we have a linear and positive operator T bounded from Y to Z, both Banach lattices, then the vector extension of T satisfies:

$$\|(\sum_{j=1}^{m}|Ty_j|^p)^{1/p}\|_Z \le \|T\|\|(\sum_{j=1}^{m}|y_j|^p)^{1/p}\|_Y \tag{2.11}$$

for every n and $1 \le p < \infty$.

(See [13], 1.d.9 for the proof, and also [12]).

The fact that ℓ^1 does not have the H.L. property can be used to show that (2.11) may fail for an operator which is only sublinear but still positive. For $p = 1$ the counterexample is simply $T = M$ with $Y = Z = L^q(\mathbf{R}^n)$, $q > 1$.

For a given $p > 1$, we can take $T(f) = (M(|f|^p))^{1/p}$ which is bounded in $L^q(\mathbf{R}^n)$ if $q > p$. However (2.11) fails with $Y = Z = L^q$ because, in this case, the left hand side of (2.11) is $\|\sum_{j=1}^n M(|f_j|^p)\|_{L^{q/p}}^{1/p}$ and in the right hand side we have $\|\sum_{j=1}^n |f_j|^p\|_{L^{q/p}}^{1/p}$, so that (2.11) would be equivalent to the boundedness of $\mathcal{M}$ in $L^{q/p}_{\ell^1}$, which does not hold. ∎

3. Some Hardy space theory

Definition 3.1: Given a Banach lattice X and a number p such that $1 \le p < \infty$, we define $\mathcal{H}^p_X(\mathbf{R}^n)$ to be the space consisting of those $f \in L^p_X(\mathbf{R}^n)$ such that for every finite set $J \subset \mathbf{Q}_+$, $\mathcal{M}_{\varphi,J}f \in L^p_X(\mathbf{R}^n)$ and $\sup_J \|\mathcal{M}_{\varphi,J}f\|_{L^p_X(\mathbf{R}^n)} < \infty$.

We endow this space with the norm:

$$\|f\|_{\mathcal{H}^p_X(\mathbf{R}^n)} = \|f\|_{L^p_X(\mathbf{R}^n)} + \sup_J \|\mathcal{M}_{\varphi,J}f\|_{L^p_X(\mathbf{R}^n)}.$$

∎

Definition 3.2: Given a Banach lattice X and a locally integrable function $f\colon \mathbf{R}^n \to X$, we define the "lattice sharp maximal function" as

$$\mathcal{M}^{\#} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

where f_Q is the vector $1/|Q| \int_Q f(y)dy$.

Of course the sup is in X and does not have to exist for all points x. We say that f is in $\mathcal{B.M.O.}_X(\mathbf{R}^n)$ if $\mathcal{M}^{\#} f \in L^{\infty}_X(\mathbf{R}^n)$. This implies, in particular, that the sup exists for a.e. x and we define the "norm" in this space as

$$\|f\|_{\mathcal{B.M.O.}_X(\mathbf{R}^n)} = \|\mathcal{M}^{\#} f\|_{L^{\infty}_X(\mathbf{R}^n)}.$$

∎

THEOREM 3.3: *For a Banach lattice X, the following conditions are equivalent:*

(1) *X satisfies the property H.L.*

(2) $\mathcal{H}^p_X(\mathbf{R}^n) = L^p_X(\mathbf{R}^n) \qquad 1 < p < \infty.$

(3) $\mathcal{H}^1_X(\mathbf{R}^n) = H^1_X(\mathbf{R}^n).$

Proof: By definition:

$$\mathcal{H}^p_X(\mathbf{R}^n) \subset L^p_X(\mathbf{R}^n).$$

Moreover, it is well known (see [3]) that for any Banach space E, the atomic Hardy space $H^1_E(\mathbf{R}^n)$ coincides with the maximal Hardy space defined as the set of those functions $f \in L^1_E(\mathbf{R}^n)$ such that $\sup_{r>0} \|\varphi_r * f(x)\|_E$ belongs to $L^1(\mathbf{R}^n)$, where φ_r is the approximate identity associated to φ smooth with compact support, as, for example, the one in definition 1.6.

Since

$$\sup_{r \in J} \|\varphi_r * f(x)\|_X \leq \|\mathcal{M}_{\varphi, J} f(x)\|_X$$

we get

$$\mathcal{H}^1_X(\mathbf{R}^n) \subset H^1_X(\mathbf{R}^n).$$

On the other hand, if X satisfies property H.L., by using property (4) in theorem 1.7, we see that

$$L^p_X(\mathbf{R}^n) \subset \mathcal{H}^p_X(\mathbf{R}^n), \qquad 1 < p < \infty$$

and, if we use (7) in the same theorem, we get:

$$H^1_X(\mathbf{R}^n) \subset \mathcal{H}^1_X(\mathbf{R}^n).$$

That way we have seen that (1) implies both (2) and (3).

For the converse implications, observe that (2) implies that property (2) in theorem 1.7 holds; but this is equivalent to H.L. Analogously (3) implies that property (7) in Theorem 1.7 holds. This is again equivalent to H.L. ∎

The following example gives an idea of the size of $\mathcal{H}^1_X$ when X fails to have the H.L. property. Of course the example is for $X = \ell^1$.

Example 3.4: For a natural number N let ℓ^1_N be the space of finite sequences $(a_j)_{j=1}^N$ of real numbers, which we view as a finite dimensional subspace of ℓ^1 by completing each vector in ℓ^1_N with infinitely many zeroes. Then

$$\bigcup_{N\geq 1} H^1_{\ell^1_N} \subset_{\neq} \mathcal{H}^1_{\ell^1} \subset_{\neq} H^1_{\ell^1}$$

Indeed, the second inclusion is proper because ℓ^1 does not satisfy H.L.

As for the first one, let $a = (a_j)_{j=1}^N$ be an atom in $H^1_{\ell^1_N}$. Then each of the componenet functions a_j is a scalar atom, and, consequently:

$$\mathcal{M}_\varphi a = \{M_\varphi a_j\}_{j=1}^N \in L^1_{\ell^1_N}$$

so that $H^1_{\ell^1_N} \subset \mathcal{H}^1_{\ell^1}$.

More concisely: $H^1_{\ell^1_N} = \mathcal{H}^1_{\ell^1_N} \subset \mathcal{H}^1_{\ell^1}$.

Now to give an example of an $f \in \mathcal{H}^1_{\ell^1}$ with infinitely many non-vanishing components, we simply have to take a scalar atom $a\colon \mathbf{R}^n \to \mathbf{R}$ and define $f(x) = (2^{-j}a(x))_j$. It is clear that $f \in \mathcal{H}^1_X$ since

$$\begin{aligned}\|\mathcal{M}_\varphi f(x)\|_{\ell^1} &= \|\{M_\varphi(2^{-j}a)(x)\}_j\| \\ &= \sum_{j=1}^{\infty} 2^{-j}|M_\varphi(a)(x)| = |M_\varphi a(x)|.\end{aligned}$$

Recall that

$$(L^p_X(\mathbf{R}^n))^* = L^{p'}_{X^*}(\mathbf{R}^n), \quad 1 \leq p < \infty \quad \text{and} \quad (H^1_X(\mathbf{R}^n))^* = B.M.O._{X^*}(\mathbf{R}^n)$$

if and only if X^* satisfies the Radon–Nikodym property (see [7] and [3]). Therefore, the following theorem is true:

THEOREM 3.5: *For a Banach lattice X, the following statements are equivalent:*

(1) *X satisfies the H.L. property and X^* satisfies the Radon-Nikodym property.*

(2) $(\mathcal{H}^p_X(\mathbf{R}^n))^* = L^{p'}_{X^*}(\mathbf{R}^n), \qquad 1 < p < \infty.$

(3) $(\mathcal{H}^1_X(\mathbf{R}^n))^* = B.M.O._{X^*}(\mathbf{R}^n).$

The next example shows that, even for such good a lattice as ℓ^2, the space $\mathcal{B.M.O.}$ is too small.

Example 3.6: $L^\infty_{\ell^2}(\mathbf{R}^n) \not\subset \mathcal{B.M.O.}_{\ell^2}(\mathbf{R}^n)$ and if we define ℓ^2_N in a similar way to ℓ^1_N in example 3.4, we have

$$\bigcup_{N\geq 1} B.M.O._{\ell^2_N} \subset_{\neq} \mathcal{B.M.O.}_{\ell^2} \subset_{\neq} B.M.O._{\ell^2}.$$

Let $f(x) = (f_j(x))_{j=1}^\infty$ be the function considered in remark 2.9. It is easy to see that for $x \in [0, 2^{-N}]$ we have $\|\mathcal{M}^\# f(x)\|_{\ell^2} \geq CN^{1/2}$ and, consequently $\mathcal{M}^\# f \notin L^\infty_{\ell^2}$ or, in other words: $f \notin \mathcal{B.M.O.}_{\ell^2}$.

Thus $L^\infty_{\ell^2} \not\subset \mathcal{B.M.O.}_{\ell^2}$.

Now if $f \in B.M.O._{\ell^2_N}(\mathbf{R}^n)$, let $f(x) = (f_j(x))_{j=1}^N$. Then

$$\|f\|_{B.M.O.} = \|f^\#\|_{L^\infty} = \sup_Q \frac{1}{|Q|} \int_Q \left(\sum_{j=1}^N |f_j(x) - (f_j)_Q|^2 \right)^{1/2} dx.$$

But

$$\begin{aligned}\|\mathcal{M}^\# f\|_{L^\infty_{\ell^2}} &= \operatorname*{ess\,sup}_x \left(\sum_{j=1}^N \left| \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f_j(y) - (f_j)_Q| dy \right|^2 \right)^{1/2} \\ &\leq \sum_{j=1}^N \|f_j^\#\|_{L^\infty} \leq N \|f^\#\|_{L^\infty}.\end{aligned}$$

On the other hand, if we have a scalar function $g \in B.M.O.(\mathbf{R}^n)$, the function $f: \mathbf{R}^n \to \ell^2$ given by

$$f(x) = \left\{ \frac{g(x)}{j} \right\}_{j=1}^\infty$$

belongs to $\mathcal{B.M.O.}_{\ell^2}(\mathbf{R}^n)$, but obviously not to any B.M.O.$_{\ell^2_N}$.

Suppose now that $f(x) = (f_j(x))_j$ is in $\mathcal{B}.\mathcal{M}.\mathcal{O}._{\ell^2}(\mathbf{R}^n)$. Then:

$$\begin{aligned}
\{\sum_{j=1}^{\infty}(f_j^{\#}(x))^2\}^{1/2} &= \{\sum_{j=1}^{\infty}(\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f_j(y)-(f_j)_Q|dy)^2\}^{1/2} \\
&= \|\mathcal{M}^{\#}f(x)\|_{\ell^2} \le \|\mathcal{M}^{\#}f\|_{L^\infty_{\ell^2}}. \\
f^{\#}(x) &= \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \|f(y)-f_Q\|_{\ell^2}dy \\
&\le \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q \|f(y)-f_Q\|_{\ell^2}^2 dy\right)^{1/2} \\
&= \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q \sum_{j=1}^{\infty}|f_j(y)-(f_j)_Q|^2 dy\right)^{1/2} \\
&\le C\sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q \sum_{j=1}^{\infty}(f_j^{\#}(y))^2 dy\right)^{1/2} \\
&\le C\|\mathcal{M}^{\#}f\|_{L^\infty_{\ell^2}}.
\end{aligned}$$

Thus

$$\|f\|_{B.M.O._{\ell^2}} = \|f^{\#}\|_\infty \le C\|\mathcal{M}^{\#}f\|_{L^\infty_{\ell^2}} = C\|f\|_{\mathcal{B}.\mathcal{M}.\mathcal{O}._{\ell^2}}.$$

Observe that an important step in the proof has been the Fefferman–Stein inequality $\int_Q |g|^2 \le C\int_Q |g^{\#}|^2$ applied to the functions $g = f_j - (f_j)_Q$ (see [10] or [11]). This can be done for sequence spaces or, in general, for Köthe function spaces.

4. The compact case

In this section our basic space will be, instead of $\mathbf{R}^n$, the torus group $\mathbf{T}$, which we shall identify in the usual way, with the compact interval [0,1]. Then Haar measure is just the ordinary Lebesgue measure in [0,1]. The funcion spaces L^p, B.M.O., etc. will be those associated with our basic measure space $\mathbf{T} = [0,1]$.

We shall be considering some non-locally-convex spaces. Recall that if $0 < r < 1$, to say that F is an r-Banach space means that we have a quasi-norm $\| \|_F$ (i.e., a functional satisfying $\|\lambda f\|_F = |\lambda|\|f\|_F$ for every $f \in F$ and every scalar λ and also

$$\|f+g\|_F \le K(\|f\|_F + \|g\|_F)$$

for every $f, g \in F$) such that $\| \|_F^r$ satisfies the triangle inequality

$$\text{(i.e. } \|f+g\|_F^r \leq \|f\|_F^r + \|g\|_F^r)$$

and F is also complete as a metric space with the distance induced by $\| \|_F^r$.

Typical examples are the Lebesgue spaces L^r and the Hardy spaces H^r, $0 < r < 1$.

For $r = 1$, r-Banach space means, of course, Banach space

Definition 4.1: Let B be a Banach space, and F an r-Banach space, $0 < r \leq 1$. W shall say that an operator T: $B \to F$ is sublinear if:

(a) $\|T(\lambda f)\|_F = |\lambda| \|T(f)\|_F$ for every $f \in B$ and every scalar λ, and

(b) $\|T(f+g)\|_F^r \leq \|T(f)\|_F^r + \|T(g)\|_F^r$.

This notion extends the more restrictive, applicable to the casse when F is an r-Banach function space, which requires instead of a) and b):

(a') $|T(\lambda f)(x)| = |\lambda||T(f)(x)|$ for a.e. x, every $f \in B$ and every scalar λ, and

(b') $|T(f+g)(x)| \leq |T(f)(x)| + |T(g)(x)|$ for a.e. x and every $f, g \in B$. ∎

Given an r-Banach function space F, we consider the space $L_F^0 = L_F^0([0,1])$ which is the space of the functions f: $[0,1] \to F$ strongly measurable, with the topology of convergence in measure (see [10] p. 529).

We have the following version of the Nikishin–Stein theorem.

THEOREM 4.2: *Let B be a Banach space consisting of functions defined on the torus with values in some Banach space. Suppose that B is invariant under translations (i.e. translations are isometries in B). Let F be an r-Banach space, and T: $B \to L_F^0$ an operator sublinear, continuous and invariant under translations. Then T is bounded from B to weak-L_F^1*

Proof: We just need to verify that Theorems 1.7, 2.4 and Corollaries 2.7 and 2.8 in chapter VI, of [10] continue to hold when we replace L^0 by L_F^0. The proofs go through without significant changes. ∎

We shall apply this theorem to the analogue for the torus of the operator $\mathcal{M}$ defined by (1.5), analogue which we shall also denote by $\mathcal{M}$. We obtain the following result:

THEOREM 4.3: *Let $0 < \alpha, \beta < 1$. Then*

(1) *$\mathcal{M}$ is bounded from $L_{L^1}^1$ to $L_{L^\alpha}^\beta$.*

(2) *$\mathcal{M}$ is bounded from $L^1_{L^1}$ to weak-$L^1_{L^\alpha}$.*

Proof: Like in Remark 1.4, when we view $\mathcal{M}$ as an operator acting on functions defined in $\mathbf{T} \times \mathbf{T}$, $\mathcal{M}$ is just the ordinary Hardy–Littlewood maximal operator M acting in the first coordinate, that is:

$$\mathcal{M}f(x,\omega) = M\{f(\cdot,\omega)\}(x).$$

But, in the case of the torus, which has finite measure, the weak type (1,1) of M is equivalent to the fact that M is bounded from L^1 to L^α for every α such that $0 < \alpha < 1$ (Kolmogorov's inequality, see [10], p. 485). Thus, for any such α fixed, we have:

$$\begin{aligned}\int_0^1 \|\mathcal{M}f(x)\|^\alpha_{L^\alpha} dx &= \int_0^1 \int_0^1 (\mathcal{M}f(x,\omega))^\alpha d\omega dx \\ &= \int_0^1 \int_0^1 (M\{f(\cdot,\omega)\}(x))^\alpha dx d\omega \\ &\le C \int_0^1 \left(\int_0^1 |f(x,\omega)| dx\right)^\alpha d\omega \\ &\le C \left(\int_0^1 \|f(x)\|_{L^1} dx\right)^\alpha,\end{aligned}$$

i.e. $\mathcal{M}$ is bounded from $L^1_{L^1}$ to $L^\alpha_{L^\alpha}$. This implies that $\mathcal{M}$ is bounded from $L^1_{L^1}$ to $L^0_{L^\alpha}$. Indeed:

$$\begin{aligned}|\{x \in [0,1]\colon \|\mathcal{M}f(x)\|_{L^\alpha} > \lambda\}| &\le \frac{C}{\lambda^\alpha} \int_0^1 \|\mathcal{M}f(x)\|^\alpha_{L^\alpha} dx \\ &\le \frac{C}{\lambda^\alpha} \|f\|^\alpha_{L^1_{L^1}}\end{aligned}$$

(see [10], p. 528 for a discussion about L^0 boundedness).

Now we can apply Theorem 4.2 to the operator $\mathcal{M}$ obtaining that it is bounded from $L^1_{L^1}$ to weak-$L^1_{L^\alpha}$, which is precisely (2). Then (1) is obtained from (2) by using Kolmogorov's inequality. ∎

Now if we start with the boundedness $\mathcal{M}\colon L^1_{L^1} \to$ weak-$L^1_{L^\alpha}$ and apply the techniques developed in the proof of Theorem 1.7, we obtain:

Corollary 4.4: *For $0 < \alpha < 1 < p < \infty$ $\mathcal{M}$ is bounded from $L^p_{L^1}$ to $L^p_{L^\alpha}$ and also $\mathcal{M}_\varphi$ is bounded from $H^1_{L^1}$ to $L^1_{L^\alpha}$ and from $L^\infty_{L^1}$ to B.M.O.$_{L^\alpha}$.*

Remark 4.5: As one would expect, the behaviour of $\mathcal{M}$ continues to be bad in L^∞. Indeed: $\mathcal{M}$ is not bounded from $L^\infty_{L^1}$ to $L^\infty_{L^\alpha}$, as the following example proves:

For N a fixed positive integer, consider the intervals

$$I_i = \left[\frac{1}{2^i}, \frac{1}{2^{i-1}}\right] \quad \text{and} \quad J_j = \left[\frac{j-1}{N}, \frac{j}{N}\right], \quad 1 \le i, j \le N,$$

and the function

$$F_N(x,\omega) = \sum_{j=1}^{N} N \mathcal{X}_{I_j}(x) \mathcal{X}_{J_j}(\omega).$$

Then

$$\begin{aligned} \|F_N\|_{L^\infty_{L^1(d\omega)}(dx)} &= \operatorname*{ess\,sup}_x \int_0^1 |F_N(x,\omega)| d\omega \\ &= \operatorname*{ess\,sup}_x \sum_{j=1}^{N} \mathcal{X}_{I_j}(x) = 1. \end{aligned}$$

On the other hand

$$\mathcal{M}F_N(x,\omega) = \sum_{j=1}^{N} N(M\mathcal{X}_{I_j}(x))\mathcal{X}_{J_j}(\omega).$$

As in Remark 2.9, we observe that

$$M\mathcal{X}_{I_j}(x) \ge 1/4 \quad \text{if } 1 \le j \le N \quad \text{and} \quad x \in [0, 2^{-N}].$$

Therefore

$$\begin{aligned} \|\mathcal{M}F_N\|_{L^\infty_{L^\alpha}} &= \operatorname*{ess\,sup}_x \left(\int_0^1 \left(\sum_{j=1}^{N} N(M\mathcal{X}_{I_j}(x))\mathcal{X}_{J_j}(\omega) \right)^\alpha d\omega \right)^{1/\alpha} \\ &= \operatorname*{ess\,sup}_x \left(\int_0^1 \sum_{j=1}^{N} N^\alpha (M\mathcal{X}_{I_j}(x))^\alpha \mathcal{X}_{J_j}(\omega) d\omega \right)^{1/\alpha} \\ &= \operatorname*{ess\,sup}_x (\sum_{j=1}^{N} N^{\alpha-1} (M\mathcal{X}_{I_j}(x))^\alpha)^{1/\alpha} \ge \frac{1}{4} N. \end{aligned}$$

∎

Theorem 4.3 and Corollary 4.4 are new for $\mathcal{M}$. The corresponding results for the conjugate function have been proved by Bourgain (see [4]) with a different method.

References

[1] C. Bennet and R. Sharpley, *Interpolation of operators,* Pure and Applied Math. 129, Academic Press, New York, 1988.

[2] J. Bourgain, *Extension of a result of Benedek, Calderón and Panzone,* Arkiv Mat. **22** (1984), 91–95.

[3] J. Bourgain, *Vector-valued singular integrals and the H^1-BMO duality,* in *Probability Theory and Harmonic Analysis* (J. A. Chao and W. A. Woyczynski, eds.), Pure and Applied Math. 98, Marcel Dekker, New York, 1986, pp. 1–19.

[4] J. Bourgain and W. J. Davis, *Martingale transforms and complex uniform convexity,* Trans. Amer. Math. Soc. **294** (1986), 501–515.

[5] D. L. Burkholder, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional,* Ann. Prob. **9** (1981), 997–1011.

[6] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. **51** (1974), 241–250.

[7] J. Diestel and J. J. Uhl, *Vector measures,* Math. Surveys, No. 15. Amer. Math. Soc., 1977.

[8] J. Duoandikoetxea, *Análisis de Fourier,* Colección Estudios, Universidad Autónoma de Madrid, 1991.

[9] C. Fefferman and E. M. Stein, *Some maximal inequalities,* Amer. J. Math. **93** (1971), 107–115.

[10] J. García-Guerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics,* North-Holland Math. Studies No. 116, 1985.

[11] J. L. Journe, *Calderón–Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón,* Lecture Notes in Mathematics **994**, Springer-Verlag, Berlin, 1983.

[12] J. L. Krivine, *Théorèmes de factorisation dans les spaces reticules,* Seminaire Maurey–Schwartz, 1973–74, Exposes 22–23, Ecole Polytechnique, Paris.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II. Function Spaces,* Springer-Verlag, Berlin, 1979.

[14] J. L. Rubio de Francia, *Martingale and integral transforms of Banach space valued functions,* Lectures Notes in Mathematics **1221**, Springer-Verlag, Berlin, 1985, pp. 195–222.

[15] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderon–Zygmund theory for vector-valued functions,* Adv. in Math. **62** (1986), 7–48.